Aggregation of information and beliefs on prediction markets with non-bayesian traders

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Abstract

Prediction markets are specific financial markets designed to produce forecasts of future events, such as political election outcomes or economic policy decisions. Empirical studies have exhibited over the years the significant accuracy of these anticipations, which tends to give credit to the *efficicent market hypothesis* advocated by the literature. However the latter relies theoretically on rational behaviors, in sharp contrast with traders' actions observed on most prediction markets. Indeed, an important fraction of participants are subject to several judgement bias. Based on Ottaviani and Sorensen's (2010) approach, we develop a framework that allows to introduce these biased traders and to study the consequences on the equilibrium properties.

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1 Introduction

Prediction markets, sometimes referred as "information markets" or "event futures", are particular financial markets where participants trade contracts whose payoff depends on the realization of unknown future events. The latter can be presidential or congressional elections, economic policy decisions (will the FED increase or decrease its interest rates for instance), but also sport events or even box-office figures. Typical contracts are *"winner-takes-all"* contracts[1], paying out a fixed amount in case of a specific event (e.g. the Republican candidate wins the US presidential election), and zero otherwise. *These markets are primarily designed to produce forecasts*, since prices on financial markets are assumed to aggregate information[2]. The final intention is to test for the different *efficient market hypotheses* introduced by Fama (1970) and stating that if traders are rational, prices convey all the relevant information about the object of trade. In this case, the price of an Arrow-Debreu asset can be interpreted as an estimate of the underlying event probability.

The first prediction market was the Iowa Presidential Stock Market (IPSM), started in 1988 by researchers[3] of the University of Iowa. Since then, it has become the Iowa Electronic Markets (IEM), the most discussed in the literature. This interest is due to the fact that, in the case of political events, one has the opportunity to compare markets' results with more conventional sources such as polls or judgement by experts. These comparisons based on two decades of research show that markets have performed remarkably well, and can even be more accurate than any other traditional tool regarding the final outcome[4]. However, if there is a consensus on the performances, there are still debates regarding the theoretical justifications[5]. While first papers were essentially motivated by the existing parimutuel betting literature[6], recent analyses have focused on the specific rules of prediction markets such as the existence of an upper bound on the amount that can be bet (Ottaviani and Sorensen, 2010), the absence of interest on the funds commited to prediction market contracts (Page and Clemen, 2010), or the typical double-auction market mechanisms used to fix prices on these markets (Serrano-Padial, 2012).

[1] Concerning political events, *seat-share* and *vote-share* contracts are also traded on prediction markets. In this case, the payoff depends, respectively, on the number of congressional or parliamentary seats allocated to parties in an election , or on the fraction of votes obtained by candidates in an election.

[2] Hayek (1945) is a forerunner of the notion of price as a signal for information: "The mere fact that there is one price for a commodity [...] brings about the solution which [...] might have been arrived at by one single mind possessing all the information which is in fact dispersed among all the people involved in the process". However note that he never alluded to the notion of *efficiency* as developed later in the literarature.

[3] Foresythe, Nelson, Neumann and Wright describe the IPSM mechanisms and its first results in *Anatomy of an Experimental Stock Market* (1992).

[4] Considering the average poll error and the average market error; Berg, Foresythe, Nelson and Rietz (2000) show that IEM outperformed polls in 9 of 15 cases between 1988 and 2000.

[5] For instance, see the "correspondence" between Manski (2004) and Wolfers and Zitzewitz (2004, 2005).

[6] Manski (2004), Gjerstad (2005) and Wolfers and Zitzewitz (2005).

Overall, the first inspiration to design prediction markets is the *Hayek hypothesis* which posits that markets can work correctly even if the participants have very limited knowledge about their environment or about other participants, due to the informational role of prices. This claim leads to the first usual assumption of prediction market models: the heterogeneity of beliefs among traders. Each agent is assumed to have a limited experience with the underlying event (which is an acceptable assumption for unique events such as political elections), and thus has different opinions.

Nevertheless, an empirical regularity, the *favorite-longshot bias*[7], tends to weaken Hayek's view and, ultimately, the efficient market hypotheses. This bias is systematically observed on betting markets, and more precisely, on parimutuel markets[8]. It is a striking result since parimutuel markets seem to be a more suited setting to achieve efficiency than the traditional financial markets, as explained by Ottaviani and Sorenson in the *Handbook of Sports and Lottery Markets* (2008)[9]. It finally becomes puzzling when one knows that prediction and parimutuel markets share broadly the same characteristics.

Prediction Markets and Judgement Bias

Several hypotheses have been developed to explain the favorite-longshot bias on parimutuel markets. Among them, risk loving bettors, market power of bookmakers or misestimation of probabilities by participants[10] . Ali (1977) provides a theoretical justification to the favorite-longshot bias solely based on heterogeneous beliefs of traders. As seen earlier, the latter is the framework used by the prediction market literature.

Parimutuel betting markets and prediction markets work similarly[11], that is why first theoretical research on prediction markets used similar frameworks (Manski (2004), Wolfers and Zitzewitz (2005), Gjerstad (2005)). These papers revise Ali's framework to study the potential extent of the favorite-longshot bias. However they do not reach a consensus on the equilibrium price properties. While Manski (2004) shows, as Ali (1977), that prices may display a bias with heterogeneous beliefs (i.e. prices do not reflect market's mean belief); the two other frameworks use different utility functions to demonstrate that prices should actually aggregate beliefs efficiently on prediction markets.

Taking into account the specific features of prediction markets, a new strand

[7] There is a favorite-longshot bias when favorites (i.e. events with short odds on betting markets) are underbet while longshots are overbet, compared to their empirical (objective) probabilities.

[8] Betting markets are markets where the money bet on all outcomes is pooled and then redistributed proportionally to winners.

[9] There are at least three reasons, presented in Section 2 of this paper.

[10] See a complete summary by Ottaviani and Sorensen in the *Handbook of Sports and Lottery Markets,* chapter 6.

[11] Parimutuel betting is observed for horse races. In this case, the money bet on all outcomes is pooled and then shared proportionally according to the odds among bettors who picked the winning outcome. Replacing bets by assets and odds by prices, one basically obtains a prediction market.

of the literature arised with Ottaviani and Sorensen. In *Aggregation of Information and Beliefs: Asset Pricing Lesson from Prediction Markets*, they offer a new approach distinct from the parimutuel betting literature. In their analysis, agents have subjective heterogeneous prior beliefs *but are also enabled to observe an objective signal (information)* correlated with the final outcome. Using a Bayesian updating process, authors show that the specific characteristics of prediction markets prevent prices from aggregating final beliefs (after observation of the signal) properly: prices underreact to information. Ultimately, they offer a new interpretation of the favorite-longshot bias. Their theoretical conclusion is still puzzling since prediction markets are used as tools to aggregate beliefs and information, and seem to work efficiently.

However, one distinction has not appeared yet in the prediction market litterature. While Ottaviani and Sorensen assume all traders assimilate perfectly information before trading, empirical studies suggest a broader spectrum of behaviors. Indeed, Foresythe et al. (1992, 2000) show evidence of the presence of *biased traders* on these markets, besides what they call *marginal traders*. Unlike biased traders, marginal traders are participants who do not have judgement bias (i.e. the case studied by Ottaviani and Sorensen). According to Foresythe et al. (1992), the diversity of participants' behaviors leads to efficient markets. Their conclusion is that rational *marginal traders* play the role of arbitrageurs, and "drive market prices, and therefore, predictions".

Based on Ottaviani and Sorensen's approach , we build a framework that allows to account for the presence of biased traders on prediction markets and study the consequences on the competitive equilibrium price.

In sequence, Section 2 provides a summary of the literature regarding prediction markets. Section 3 presents the framework used to introduce biased traders and displays the main results concerning the competitive equilibrium. Then, Section 4 shows the necessary extensions of the model and Section 5 concludes. The appendix collects computations and proofs.

2 Literature Review

We differentiate two strands of the literature regarding prediction markets, the first one being a new interpretation of the existing parimutuel betting framework and the second, an analysis of the particular features of prediction markets and their consequences on the aggregation of information and beliefs.

Parimutuel Betting

Since both trading process are similar, first papers on prediction markets find support in the parimutuel betting literature[12]. Bets are replaced by Arrow-Debreu assets and odds by prices on prediction markets. Parimutuel markets,

[12]Some papers quote Hayek to justify market's efficiency but use static models from the parimutuel betting literature, which may seem unadapted since the Hayek hypothesis precisely

due to their specific features, have led to a wide empirical and theoretical literature. Indeed, as shown by Ottaviani and Sorensen (2008), these markets are well suited to study market efficiency. First, there is a pre-specified termination point at which asset's (bet's) value becomes certain, whereas a security on traditional stock market is infinitely lived, making its valuation by traders more complex. Secondly, in contrast with usual financial markets, the realized outcomes are exogeneous and independant of the prices and trading process on parimutuel markets[13]. Finally, prices on these markets are not set by market makers, avoiding potential mispricing.

Ali (1977) analyzes parimutuel betting over 20,247 horse races and uncovers a systematic "anomaly" regarding efficiency which is the favorite-longshot bias. Introducing bettors with heterogeneous beliefs, he retrieves this bias in a theoretical approach.

Assume a continuum of risk neutral traders noted i with belief q_i concerning the winning of a particular horse. Assume also that beliefs are distributed according to a continuous and strictly increasing cumulative distribution function $G(.)$. In parimutuel markets, *the subjective* or *market-implied probability* of an event realization is defined by the proportion of the pool bet on it, noted m. *The objective* or *empirical probability* in Ali (1977), is defined by the median belief of the distribution $G(.)$. Obviously, there is a bias when objective and subjective probabilities are not concordant.

Intuitively, a bettor bets on an event realization when his or her own belief q_i is greater than the market probability m[14]. Since in Ali's model, all bettors wager the same amount, the proportion of bettors with a belief greater than the market probability defines precisely the share of the total pool bet on the event realization (i.e **the market probability**). Hence,

$$1 - G(m) = m$$

If the median of the belief distribution (i.e. **the objective probability**) is γ, then by definition

$$1 - G(\gamma) = \frac{1}{2}$$

Since G is strictly increasing we observe that:

- if $m > \frac{1}{2}$ (i.e. the event is a market's favorite), then $\gamma > m$ (i.e. the market probability is lower than the empirical probability defined by the median belief).

- Similarly, if $m < \frac{1}{2}$ (i.e. the event is a market's longshot) then $\gamma < m$ (i.e. the market probability is greater than the empirical probability).

rests on "some prior groping process of market discovery " and would rather suggest a dynamic approach.

[13]This assumption is relaxed for prediction markets in Lieli and Nieto-Barthaburu (2009) where an exogeneous agent can alter the event probability after observing market prices.

[14]One finds easily this result with a wealth-maximization program (agents are risk-neutral)

There is a favorite-longshot bias: favorites are underbet compared to the objective probability and longshots are overbet.

Manski (2004) is the first to provide a theoretical framework for prediction markets, relying mainly on Ali's work. In his model, the market-implied probability becomes the price of an Arrow-Debreu asset that pays out one in case of a particular event and zero otherwise. As in the seminal model, he shows that prices may not reflect the market's average belief and concludes logically by stating that prediction markets can not be used as tools to aggregate beliefs.

Gjerstad (2005) and Wolfers and Ziztewitz (2005) respond by introducing risk aversion in Ali's framework. Using a logarithmic utility function[15], they show that, for any belief distribution, price is actually equal to the market's mean belief on prediction markets with Arrow-Debreu assets. Thus, more empirical studies on prediction markets are still necessary in order to evaluate the degree of risk aversion of traders.

For the first time, Page and Clemen (2010) introduce a time dimension in Ali's framework. Indeed, prediction markets are typically started several months before the event outcome and traders receive no interest on the funds committed to prediction market contracts. Assuming time discounting preferences, it means that a cost is induced. As a result, agents with a belief close to the market price abstain from trading, since the cost is more likely to overcome the expected gains. The abstention is even more pronounced for the favorites because of their higher prices, that require a higher commitment and thus, a larger cost due to traders' preference for the present. Finally, authors offer a new theoretical explanation of the favorite-longshot bias and cast doubt on the ability of prediction markets to forecast long-term events.

Introduction of Information

Empirical studies suggest discrepancies between market-implied and empirical probabilities and all the theoritical prediction market frameworks derived from Ali's work attempt to find theoretical foundations with the following methodology: the objective probability is determined by the median or average belief and is compared to the subjective probability defined by the asset price. However, considering the median belief as the empirical probabilitiy in prediction market models implies whether that:

- Traders are a representative sample of the overall population. For instance, in the case of political elections, it would mean that traders on prediction markets have the same presidential preferences than the total population of voters, and in the same proportions. Empirical studies do not give support to this claim[16].

[15]A logarithmic utility function implies a coefficient of relative risk aversion equals to 1, which is a typical estimate in empirical studies (for instance, Holt and Laury, 2002, *Risk Aversion and Incentive Effects*).

[16]See Foresythe et al. (1992, 2000).

- Traders have access to objective *information* about the outcome and update their beliefs. In this case it would mean that prices, by aggregating updated beliefs, contain information about the final outcomes. Then, rational traders as in Ali's model, should be able to extract information from the price and modify their demand. It is actually an outcome from the critique of the Walrasian approach to the price formation with heterogeneous beliefs presented by Grossman (1989).

The latter remark is the cornerstone of Ottaviani and Sorensen's framework. Contrary to previous papers, they introduce explicitly information in their framework with a signal. Then they assume a bayesian update of beliefs and analyze how price aggregates these final beliefs. They do not define an ad hoc endogeneous objective probability, but rather study how price is related with the exogenous information observed by traders. Therefore, they depart themselves from Ali's model and become immune to the critique mentioned above. One of the main conclusions is that, even if all traders observe the same signal and agree on its interpretation, the price underreacts to information (i.e. the price fails to aggregate properly final beliefs).

Their result relies partly on a particular feature of prediction markets which is the upper bound on the amount that can be invested[17], which generates a *wealth effect*. Associated with heterogeneous *prior* beliefs and access to information, authors show that markets can not be perfectly efficient. These three "ingredients" (heterogeneous prior beliefs, information and wealth effect) are necessary to the underreaction result.

Indeed, Varian (1985) already combined heterogeneous beliefs and information but did not introduce a wealth effect. Likewise, Ottaviani and Sorensen show with an extension of a classical Rubinstein's (1974) result that access to information and wealth effect, with a common prior, do not necesarily lead to underreaction. The third possibility, namely, heterogeneous beliefs and wealth effect without information, is Ali's case with its weaknesses presented above.

Biased Traders

Note that traders are assumed perfectly rational in Ottaviani and Sorensen's framework. It means traders agree on the interpretation of information and update their beliefs in a Bayesian rational way. Forsythe et al. (1992), thanks to individual data of IEM, show that a majority of traders[18] are actually subject to numerous judgement bias that prevent them to interpret information in a Bayesian way. Still, theoretical prediction market frameworks do not account for different types of traders[19]. To our knowledge, only Serrano-Padial (2012) considered a similar case with naive and sophisticated traders (but in a double-auction market), and eventually displayed a favorite-longshot bias. Hereafter,

[17] For instance, traders on IEM are authorized to wager up to $500.

[18] Forsythe et al. (1992), with specific criteria, exhibit only 22 marginal (rational) traders out of 192 participants on the 1988 US presidential election market of the IEM

[19] Studies of traditional financial markets have already accounted for biased traders. See for instance Lam, Liu and Wong (2009) for pseudo-bayesian traders.

using Ottaviani and Sorensen's model, we build a framework that allows for the introduction of biased traders and derive the Walrasian equlibrium price properties.

3 The Framework[20]

Prediction Market

Following Ottaviani and Sorensen, we consider a binary prediction market in which traders can take positions on the realization of an event E, or on the complementary event E^c. There are two Arrow-Debreu assets in this market:

- One asset that pays out 1 currency unit if event E is realized, 0 otherwise.
- One asset that pays out 1 currency unit if event E^c is realized, 0 otherwise.

The total supply of each asset is normalized to 1.

Trader i enters the market by obtaining an equal number of both assets, w_{i0} units. This is a particular feature of the IEM that ensures a zero-sum game[21]. Participants have an upper bound on the amount they are allowed to invest on this market and are not authorized to hold a negative number of asset (no "short sales"). We normalize the sum of the two prices to one. Hereafter, we focus on the price p of the asset paying in event E, since the other case is symmetrical.

Traders

We assume *heterogeneous beliefs* among participants regarding the realization of event E. Therefore, traders have at first different subjective *prior* beliefs q_i.

Unlike previous frameworks, Ottaviani and Sorensen's model introduces explicitly objective information through the observation of a public signal s by each participant before trading. Density of the signal conditional to the state $\omega \in \{E, E^c\}$ is $f(s \mid \omega)$. Thus, the likelihood ratio for signal realization is defined by $L(s) = \frac{f(s|E)}{f(s|E^c)}$. The signal is never fully state-revealing, then $L(s) \in (0, \infty)$. Obviously, a signal realization s in favor of event E (resp. E^c) has a likelihood ratio greater than 1 (resp. lower than 1). The observation of the signal leads each trader to update his or her *prior* belief q_i to a subjective *posterior* belief u_i. The key point of the paper is to determine a suitable *updating rule* of beliefs.

We assume a continuum of *risk-neutral* traders that maximize their expected wealth $u_i w_i(E) + (1 - u_i) w_i(E^c)$ where $w_i(E)$ is the quantity of asset that pays out in E (resp. E^c) eventually held.

In order to define an aggregate demand in the following, the initial distribution of assets over individuals is represented by the cumulative distribution

[20] Troughout the remaining, "*" sends the reader to the appendix

[21] As specified in its rules, the IEM do not make profits or losses, since these markets are designed for educational purposes only

function F where $F(q)$ is the initial share of assets held by all traders with a belief lower than q. F is assumed to be continuous and strictly increasing on the interval where $F \notin \{0, 1\}$

3.1 Determination of Posterior Beliefs

3.1.1 Bayesian Updating Rule

In the seminal model, all traders agree on the interpretation of the signal and update in a Bayesian rational way. Thus, according to Bayes' rule, their posterior belief is defined by:

$$\frac{\pi_i}{1-\pi_i} = \frac{q_i}{1-q_i} L(s) \tag{1}$$

where π_i is the subjective *posterior* belief with a bayesian updating rule.

3.1.2 Alternative Updating Rule

Empirical studies on prediction markets suggest various behaviors regarding the interpretation of information. In the first analysis of the IEM, Foresythe et al. (1992) highlight the existence of biased traders and discuss the potential nature of the bias by using arguments from both psychologists and political scientists.

The first bias documented is the ***assimilation-contrast effect***[22] that states that "an individual's preference for an outcome biases his or her interpretation of information about the likelihood of the outcome occuring". The second bias mentioned is called the ***false-consensus effect***[23] and posits that traders "tend to overestimate the extent to which their views are representative of the population"[24]. Both effects show the importance of prior belief in traders' estimates of the event's probability.

We account for this facts by using an updating rule distinct from the bayesian one. Assume posterior beliefs are then defined by:

$$\delta_i(L) = (1-\alpha)\pi_i(L) + \alpha q_i$$

where $\alpha \in (0, 1)$ is constant among traders and where δ_i is the subjective posterior belief with this different update rule. Note that, by definition, $\delta_i \in (min(q_i, \pi_i), max(q_i, \pi_i))$.

Independently of the prediction market literature, Epstein, Noor and Sandroni (2009) present this alternative to the bayesian benchmark, based on the

[22] Literature mentioned by the original paper: Sherif and Hovland, *Social Judgement: Assimilation and Contrast Effects in Communication and Attitude Change* (1961); Parducci and Marshall, *Assimilation vs. Contrast in the Anchoring of Perceptual Judgements of Weights* (1962).

[23] Literature mentioned by the original paper: Ross et al., *The False Consensus Effect: An Egocentric Bias in Social Perception and Attribution Processes* (1977) and Brown, *A False Consensus Bias in 1980 Presidential Preferences* (1982).

[24] Both quotations in this paragraph are from Foresythe et al. (1992)

work of behavioral economists [25]. They also rely on the idea that Bayesian updating rules may be costly so that, even if agents would prefer to be bayesian, they may use simpler non-bayesian updates. With this updating rule, individuals place excessive weights on prior beliefs and underreact to new information, in line with the two bias mentioned above. We turn now to the determination of the equilibrium price.

3.2 Competitive Equilibrium Price With Non Bayesian Agents

In this section, we provide the main steps leading to the determination of the equilibrium price p of the E asset.

First, we need to define risk neutral traders' demand for each asset. Assume participants observe a signal realization with likelihood L that leads to different updated beliefs $u_i \in \{\pi_i, \delta_i\}$. Since we assume risk neutrality, agents either buy only E assets or E^c assets. The subjective expected return on *one* E asset at a price p is defined by:

$$u_i(1-p) - (1-u_i)p = u_i - p$$

Similarly, the expected return of one E^c asset is $(1-u_i)p - u_i(1-p) = p - u_i$[26].

Thus:

- if $u_i > p$, traders exchange their entire endowment of the E^c asset at price $1-p$ in order to buy $\frac{w_{i0}(1-p)}{p}$ units of the E asset. Their final portfolio is then $\frac{w_{i0}}{p}$ units of the E asset and 0 unit of the E^c asset.
- Conversely, if $u_i < p$, agents exchange their entire endowment of the E asset to buy the E^c asset. Their final portfolio becomes $\frac{w_{i0}}{1-p}$ units of the E^c asset and 0 unit of the E asset.
- In the specific case $u_i = p$, individuals are indifferent between any trade.

Considering Bayesian updates, Ottaviani and Sorensen show that the equilibrium price is the unique solution to the equation:

$$p = 1 - G\left(\frac{p}{(1-p)L + p}\right) \tag{2}$$

and is strictly increasing in L.

Hereafter, we provide a solution for the updating rule leading to the posterior belief $u_i = \delta_i$ (alternative non-bayesian update).

[25] For instance, Kahneman and Tversky (1974), *Judgement under uncertainty: heuristics and biases,* Science 185

[26] Remember that the price of the E^c asset is $1-p$

Equilibrium Price

For a given likelihood ratio L, we have $\delta_i = (1-\alpha)\pi_i + \alpha q_i$. From (1), one shows that $\pi_i = \frac{q_i L}{(L-1)q_i+1}$. We rewrite:

$$\delta_i = (1-\alpha)\frac{q_i L}{(L-1)q_i+1} + \alpha q_i$$

Agents buy the E asset if $\delta_i > p$ or similarly:

$$(1-\alpha)\frac{q_i L}{(L-1)q_i+1} + \alpha q_i > p$$

This inequation becomes:

$$q_i^2(\alpha(L-1)) + q_i\left[L + (\alpha+p)(1-L)\right] - p > 0$$

We need to *infer* from this inequation the threshold value $q^*(L,p,\alpha)$ above which traders would buy the E asset. Note that setting $\alpha = 0$, one obviously retrieves the threshold defined by the original paper with bayesian updates.

Computations* provide the following result:

$$q^*(L,p,\alpha) = \frac{\sqrt{\left[L+(\alpha+p)(1-L)\right]^2 + 4\alpha(L-1)p} - \left[L+(\alpha+p)(1-L)\right]}{2\alpha(L-1)} \quad (3)$$

Therefore, for a given L and α, every trader with a posterior belief $\delta_i > p$, or equivalently $q_i > q^*$, demands the E asset in amount $\frac{w_{i0}}{p}$. Using the cumulative distribution function, the aggregate demand becomes:

$$\frac{1-F(q^*)}{p}$$

where $1 - F(q*)$ represents the total initial asset endowment of participants with a prior belief $q_i > q*$. Aggregate supply in our framework is equal to 1. Thus, at equilibrium:

$$\frac{1-F(q^*)}{p} = 1$$

Finally, the competitive equilibrium price is defined by the following equation:

$$p = 1 - F(q^*)$$

Proposition 1* *For $L \neq 1$, the competitive equilibrium price, p, with non bayesian traders is the unique solution to the equation*

$$p = 1 - F\left(\frac{\sqrt{[L + (\alpha + p)(1 - L)]^2 + 4\alpha(L - 1)p} - [L + (\alpha + p)(1 - L)]}{2\alpha(L - 1)}\right)$$

and is a strictly increasing function of the information realization L and a strictly decreasing function of α.

3.3 Underreaction of Price to Information

One of the main achievements of the original framework is to exhibit an underreaction of the price to new information *compared* to the bayesian reaction of traders' beliefs. Even if participants update according to Bayes' rule and interpret the information in the same way (i.e. agree on the conditional density of the signal $f(s \mid \omega)$), price does not reflect this agreement. Then, in contradiction with the Arrow-Debreu assets literature, *market's expectations and price differ.* Ultimately, authors provide a new explanation for market inefficiencies such as the favorite-longshot bias without relying on an ad hoc definition of the event's *objective/empirical probability*

3.3.1 Underreaction of Price with Bayesian Updates

Authors illustrate underreaction in their framework with the following example. Assume traders observe a public signal that delivers information more favorable to event E (i.e. an increase in L). According to (2), price of the E asset, p, is higher when L increases. It implies that traders willing to purchase the E asset can buy fewer units since the bound $\frac{w_{i0}}{p}$ is decreasing in p, there is a *wealth effect*. Conversely, traders who wish to buy the E^c asset at price $1 - p$ purchase more units. Therefore, with a higher L, optimists about the event E buy less, while pessimists (who sell the E asset) sell more. As a result, if all traders who were buying the E asset before the increase in L were still purchasing the same asset after observation of the signal, there would be an insufficient demand for E. Equivalently, there would be an excess demand for the E^c asset. For the market to clear, some pessimistic traders must move to the optimistic side. Then the marginal trader that determines the price holds a pessimistic belief about the event realization. Due to this negative effect, the price moves slower than posterior beliefs of traders.

It is a puzzling result since these markets are precisely designed to aggregate information and beliefs.

Formally, for two different information realizations L and $L' > L$, Ottaviani and Sorensen displays the underreaction of price according to the following equation[27]:

[27] The proof provided in the original paper

$$log\left(\frac{\pi(L')}{1-\pi(L')}\right) - log\left(\frac{\pi(L)}{1-\pi(L)}\right) > log\left(\frac{p(L')}{1-p(L')}\right) - log\left(\frac{p(L)}{1-p(L)}\right) \quad (4)$$

In our framework, the logarithm transformation of the function $\frac{\delta(L)}{1-\delta(L)}$ do not allow to conclude regarding underreaction of price, hence we move from a discrete time to a continuous time setting.

3.3.2 Reaction of Price with Non Bayesian Updates

Hereafter, in order to simplify computations, we consider the specific case of a uniform distribution of initial endowments, i.e. $F(q^*) = q^*$. Thus the competitive equilibrium price equation given by Proposition 1 becomes:

$$p = 1 - q*$$

where $q* = \frac{\sqrt{[L+(\alpha+p)(1-L)]^2+4\alpha(L-1)p} - [L+(\alpha+p)(1-L)]}{2\alpha(L-1)}$

Proposition 2* *If we assume a uniform distribution, the competitive equilibrium price implicitly defined by Proposition 1 is:*

$$p = \frac{\alpha(L-1) + 2L - \sqrt{[\alpha(1-L) - 2L]^2 - 4L(L-1)(1+\alpha)}}{2(L-1)(1+\alpha)}$$

Underreaction

We want to verify if underreaction, as defined by (3), also occurs with non-bayesian traders. Observe that (3) is true for any $L' > L$. Then we define $L' = L + dL$ with $dL > 0$. We rewrite (3):

$$\frac{log\left(\frac{\delta(L+dL)}{1-\delta(L+dL)}\right) - log\left(\frac{\delta(L)}{1-\delta(L)}\right)}{L+dL-L} > \frac{log\left(\frac{p(L+dL)}{1-p(L+dL)}\right) - log\left(\frac{p(L)}{1-p(L)}\right)}{L+dL-L}$$

Taking the limits, one obtains:

$$\lim_{dL\to 0^+} \frac{log\left(\frac{\delta(L+dL)}{1-\delta(L+dL)}\right) - log\left(\frac{\delta(L)}{1-\delta(L)}\right)}{dL} > \lim_{dL\to 0^+} \frac{log\left(\frac{p(L+dL)}{1-p(L+dL)}\right) - log\left(\frac{p(L)}{1-p(L)}\right)}{dL}$$

which is equivalent to:

$$\frac{\partial log\left(\frac{\delta(L)}{1-\delta(L)}\right)}{\partial L} > \frac{\partial log\left(\frac{p(L)}{1-p(L)}\right)}{\partial L}$$

it leads to:

$$\frac{\frac{\partial\left(\frac{\delta(L)}{1-\delta(L)}\right)}{\partial L}}{\frac{\delta(L)}{1-\delta(L)}} > \frac{\frac{\partial\left(\frac{p(L)}{1-p(L)}\right)}{\partial L}}{\frac{p(L)}{1-p(L)}}$$

meaning that if the growth rate of $\frac{\delta(L)}{1-\delta(L)}$ is greater than the growth rate of $\frac{p(L)}{1-p(L)}$, there is an *underreaction of price to information.*

One can also show easily that $\frac{\partial log\left(\frac{\delta(L)}{1-\delta(L)}\right)}{\partial L} = \frac{\frac{\partial\delta(L)}{\partial L}}{\pi(L)(1-(L))}$ and $\frac{\partial log\left(\frac{p(L)}{1-p(L)}\right)}{\partial L} = \frac{\frac{\partial p(L)}{\partial L}}{p(L)(1-p(L))}$, which leads to the following proposition:

Proposition 3* *For every $L > 0$, Ottaviani and Sorensen's underreaction of price is also defined in continous time by the inequality:*

$$\frac{\frac{\partial\delta(L)}{\partial L}}{\frac{\partial p(L)}{\partial L}} > \frac{\delta(L)(1-\delta(L))}{p(L)(1-p(L))}$$

With non bayesian updates and a uniform distribution of initial wealth, this inequality does not hold; meaning that underreaction of price, as defined in Ottaviani and Sorensen's framework, is not confirmed. The price does not necessarily underreact to the interpretation of information made by non-bayesian traders.

4 Extensions

Different Updating Rules

We have presented a framework that allows for the introduction of traders who are not perfectly rational since they do not update in a Bayesian rational way after observing a public signal. However, the competitive equilibrium derived account for a constant $\alpha > 0$, meaning that all traders underreact in the same way to the signal. The next step would be to introduce agents with different

updating rules α_i. Nevertheless, several issues must be solved before deriving the equilibrium.

First, the introduction of traders' dependent updating rules implies to define a new distribution function. For instance $P(q,\alpha)$ would be the initial cross-sectional distribution of beliefs and updating rules. A first step to solve the model would be to determine whether prior beliefs and updating rules are statistically independent. Empirical studies are needed to see how traders interpret information depending on the value of their prior beliefs.

A way to avoid this issue would be to assume a common *prior* belief and the following updating rule:

$$\delta_i = (1-\alpha_i)\pi + \alpha_i q$$

But then all beliefs belong to the narrow interval: $[min(q,\pi), max(q,\pi)]$

Secondly, it is shown empirically in the financial literature that investors pay too much attention to extreme information[28], meaning that the updating rule may also depend on the quality of information.

Dynamic Model

In Ottaviani and Sorensen's framework, the model and rationality of traders are common knowledge. Thus, all the information is revealed at the equilibrium (i.e. $p(L)$ is injective). Therefore, in the dynamic setting, there is no exchange when information appears after the initial round of trade; as suggested by the *no trade* theorem demonstrated by Milgrom and Stokey (1982). One possibility to bypass the *no trade* theorem in our framework is to assume time-variant updating rule. Lam, Liu and Wong (2009) present, for a traditional stock market, a pseudo-bayesian approach with time-variant weights on the information. Under this consideration, they can model two typical behavioral biases observed on traditional financial markets: traders' usage of conservatism heuristics (underreaction to information) or representativeness heuristics (overreaction to information).

5 Conclusion

Empirical studies suggest the presence of at least two types of traders on prediction markets: *rational/marginal traders* and *biased traders*. Based on Ottaviani and Sorensen's framework, we build a prediction market model that allows to introduce non-bayesian traders. In this case, we show that equilibrium price properties differ from the seminal model ones. In particular, we cannot ascertain that price underreacts to the interpretation of information made by biased-traders. In order to exhibit new price patterns, the next step would be to allow for different type of traders (biased/unbiased) and a dynamic setting.

[28]Griffin and Tversky (1992) in "The Weighing of Evidence and the Determinants of Confidence", *Cognitive Psychology,* 24, 411–435.

6 Appendix

We provide the main steps of the computations that led to the different propositions

Determination of the threshold q^*

- For the case $L = 1$, the solution is straighforward, using (2) we have $q^* = p$, as in Ottaviani and Sorensen. Then one has an equilibrium price defined by the equation $p = 1 - F(p)$

- For the case $L \neq 1$, we can compute the threshold value of q_i by rewritting:

$$q_i^2(\alpha(L-1)) + q_i\left[L + (\alpha+p)(1-L)\right] - p = 0 \tag{5}$$

We recognize a quadratic function of q_i with discriminant Δ such that:

$$\Delta = \left[\, L + (\alpha+p)(1-L)\right]^2 + 4\alpha(L-1)p$$

To determine the sign of this discriminant, we rewrite:

$$\Delta = \left[1-(\alpha+p)\right]^2 L^2 + 2\left[\alpha(1-\alpha)+p(1-p)\right]L + (\alpha-p)^2 \tag{6}$$

For $\alpha + p = 1$, it can be shown that $\Delta > 0$

When $\alpha + p \neq 1$, it is a new quadratic function of L with a new discriminant Δ_Δ defined by:

$$\Delta_\Delta = 4\left[\alpha(1-\alpha)+p(1-p)\right]^2 - 4\left[1-(\alpha+p)\right]^2(\alpha-p)^2$$

Δ_Δ is positive if:

$$\mid \alpha - \alpha^2 + p - p^2 \mid > \mid 1-(\alpha+p) \mid\mid \alpha - p \mid$$

One shows this inequality holds for $p \in (0,1)$ and $\alpha \in (0,1)$. Then $\Delta_\Delta > 0$ under these conditions. Thus, we derive two roots:

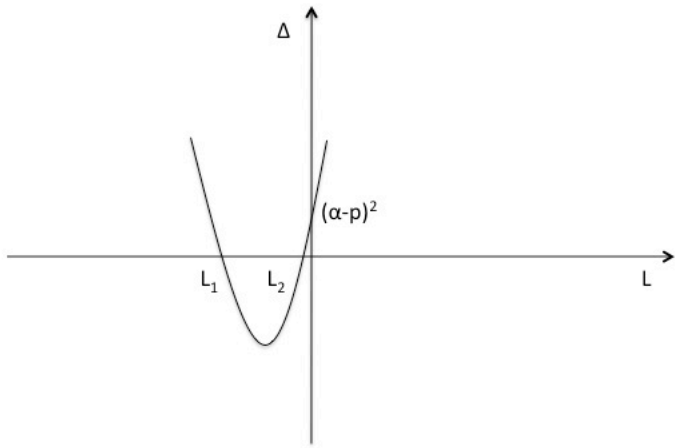
$$L_1 = \frac{-\left[\alpha(1-\alpha)+p(1-p)\right] - \sqrt{\Delta_\Delta}}{\left[1-(\alpha+p)\right]^2}$$

$$L_2 = \frac{-\left[\alpha(1-\alpha)+p(1-p)\right] + \sqrt{\Delta_\Delta}}{\left[1-(\alpha+p)\right]^2}$$

It can be shown that $L_1 < 0$ and $L_2 < 0$

We proceed to a simple graphic interpretation to determine the sign of Δ. First, observe that the coefficient of L^2 in (5) is $\left[1-(\alpha+p)\right]^2$ wich is positive under our asumption $\alpha + p \neq 1$. It means Δ is a parabola that opens upward. Second, for $L = 0$, Δ value is $(\alpha - p)^2 \geq 0$. We can conclude that, for any $L \in (0, \infty)$, $\Delta > 0$ when $\alpha + p \neq 1$ (see Figure 1).

Figure 1



Therefore, for both cases $\alpha + p = 1$ and $\alpha + p \neq 1$, we define 2 solutions q_i^1 and q_i^2 of (3):

$$q_i^1 = \frac{-[L + (\alpha + p)(1 - L)] - \sqrt{\Delta}}{2\alpha(L - 1)}$$

$$q_i^2 = \frac{-[L + (\alpha + p)(1 - L)] + \sqrt{\Delta}}{2\alpha(L - 1)}$$

The solution must satisfy $0 < q^* < 1$. We have to distinguish 2 cases: $L < 1$ and $L > 1$. We provide a summary of the different signs of q_i depending on the value of L (all the computations to find the differents results presented in the table are not written here):

L<1	L>1
$-[(L+(\alpha+p)(1-L)] - \sqrt{\Delta} < 0$	$-[(L+(\alpha+p)(1-L)] - \sqrt{\Delta} < 0$
$2\alpha(L-1) < 0$	$2\alpha(L-1) > 0$
$\mathbf{q_i^1 > 1}$	$\mathbf{q_i^1 < 0}$
$-[(L+(\alpha+p)(1-L)] + \sqrt{\Delta} < 0$	$-[(L+(\alpha+p)(1-L)] + \sqrt{\Delta} < 0$
$2\alpha(L-1) < 0$	$2\alpha(L-1) > 0$
$\mathbf{0 < q_i^2 < 1}$	$\mathbf{0 < q_i^2 < 1}$

Thus, the only threshold value of q_i for any $L \in (0, \infty)$ is (1).

Proof of Proposition 1

We only provide here a proof for the case $L > 1$.

Price is defined implicitly by Proposition 1. Note that the left hand side is strictly increasing with p and, naturally, is equal to 0 when $p = 0$ and 1 when $p = 1$. For the right hand side, one can easily show that, for any α, $q^* = 0$ with $p = 0$ and $q* = 1$ with $p = 1$. Then $1 - F(q^*) = 1$ in $p = 0$ and $1 - F(q*) = 0$ in $p = 1$. Then, let's study:

.

$$\frac{\partial q*}{\partial p} = \frac{1}{2\alpha(L-1)} \left\{ [2(1-L)[L+(\alpha+p)(1-L)+4\alpha(L-1)] \frac{1}{2\sqrt{\Delta}} - (1-L) \right\}$$

We focus on the sign of the expression in brackets:

$$(1-L)[L+(\alpha+p)(1-L)-2\alpha(1-L)]\frac{1}{\sqrt{\Delta}} - (1-L)$$

$$\Leftrightarrow (1-L)\left[\frac{L+(\alpha+p)(1-L)-2\alpha-\sqrt{\Delta}}{\sqrt{\Delta}}\right]$$

which is positive, using the fact that $q^* > 0$, $L+(\alpha+p)(1-L)-2\alpha-\sqrt{\Delta} < 0$ when $L > 1$. Thus:

$$\frac{\partial q*}{\partial p} = \frac{(1-L)\left[\frac{L+(\alpha+p)(1-L)-2\alpha-\sqrt{\Delta}}{\sqrt{\Delta}}\right]}{2\alpha(L-1)} > 0$$

Since the function F is strictly increasing, the right hand side is strictly decreasing in p while the left hand side is strictly increasing. **Thus the competitive equilibrium price defined implicitly by Proposition 1 is unique.**

Next, *we prove that the equilibrium price defined implicitly by Proposition 1 is an increasing function of* L, i.e. when traders have more information favorable to event E, the price of the E asset increase.

Observe that when L rises, the left hand side of the equilibrium price equation is unaffected. Concerning the right hand side, let's compute:

$$\frac{\partial q*}{\partial L}=\frac{\left\{\frac{1}{2}\left[2(1-(\alpha+p))[L+(\alpha+p)(1-L)+4\alpha p\right]\frac{1}{\sqrt{\Delta}}-[1-(\alpha+p)]\right\}2\alpha(L-1)-\left\{\sqrt{\Delta}-[L+(\alpha+p)(1-L)]\right\}2\alpha}{[2\alpha(L-1)]^2}$$

$$\Leftrightarrow \frac{\partial q*}{\partial L}=\frac{\left\{[(1-(\alpha+p))[L+(\alpha+p)(1-L)+2\alpha p]\frac{1}{\sqrt{\Delta}}-[1-(\alpha+p)]\right\}2\alpha(L-1)-\left\{\sqrt{\Delta}-[L+(\alpha+p)(1-L)]\right\}2\alpha}{[2\alpha(L-1)]^2}$$

Let's study the numerator:

$$2\alpha\left\{\frac{(L-1)(1-(\alpha+p))[L+(\alpha+p)(1-L)]}{\sqrt{\Delta}}+\frac{2\alpha p(L-1)}{\sqrt{\Delta}}-[1-(\alpha+p)](L-1)-\sqrt{\Delta}+[L+(\alpha+p)(1-L)]\right\}$$

$$\Leftrightarrow 2\alpha\left\{\frac{(L-1)(1-(\alpha+p))[L+(\alpha+p)(1-L)]+2\alpha p(L-1)-[1-(\alpha+p)](L-1)\sqrt{\Delta}-\Delta+\sqrt{\Delta}[L+(\alpha+p)(1-L)]}{\sqrt{\Delta}}\right\}$$

Let's study the numerator:

$$(L-1)(1-(\alpha+p))[L+(\alpha+p)(1-L)]+2\alpha p(L-1)-\Delta+\sqrt{\Delta}\left[L+(\alpha+p)(1-L)-\{L-1-(\alpha L-\alpha+pL-p)\}\right]$$

$$\Leftrightarrow (L-1)(1-(\alpha+p))[L+(\alpha+p)(1-L)]+2\alpha p(L-1)-[L+(\alpha+p)(1-L)]^2-4\alpha(L-1)p+\sqrt{\Delta}$$

$$\Leftrightarrow (L-1)(1-(\alpha+p))[L+(\alpha+p)(1-L)]-2\alpha p(L-1)-[L+(\alpha+p)(1-L)]^2+\sqrt{\Delta}$$

$$\Leftrightarrow [(L-1)(1-(\alpha+p))-L-(\alpha-L\alpha+p-pL)]\,[L+(\alpha+p)(1-L)]-2\alpha p(L-1)+\sqrt{\Delta}$$

$$\Leftrightarrow \sqrt{\Delta}-[L+(\alpha+p)(1-L)]-2\alpha p(L-1)$$

So finally:

$$\frac{\partial q^*}{\partial L}=\frac{2\alpha\left[\frac{\sqrt{\Delta}-[L+(\alpha+p)(1-L]-2\alpha p(L-1)}{\sqrt{\Delta}}\right]}{[2\alpha(L-1)]^2}$$

Let's study the sign of this expression by studying the sign of $\sqrt{\Delta}-[L+(\alpha+p)(1-L)]-2\alpha p(L-1)$. We have:

$$\sqrt{\Delta}-[L+(\alpha+p)(1-L)]-2\alpha p(L-1)>0$$

$$\Leftrightarrow \sqrt{\Delta}-[L+(\alpha+p)(1-L)]>2\alpha p(L-1)$$

since $L>1$, we have

$$p < \frac{\sqrt{\Delta} - [L + (\alpha + p)(1 - L)]}{2\alpha(L - 1)}$$

Or equivalently:

$$p < q^*$$

Simulations show $q^* < p$ for any $L > 1$

Then:

$$\frac{\partial q^*}{\partial L} < 0$$

Since F is a strictly increasing function , when L rises, the right hand side is strictly increasing. **Ultimately, the equilibrium price is an increasing function of L.**

Finally, we study the derivative of q^* with respect to α:

$$\frac{\partial q*}{\partial \alpha} = \frac{\left\{\frac{1}{2}\left[2(1-L)[L + (\alpha + p)(1 - L)] + 4(L-1)p\right]\frac{1}{\sqrt{\Delta}} - (1-L)\right\} 2\alpha(L-1) - \left[\sqrt{\Delta} - [L + (\alpha + p)(1 - L)\right] 2(L-1)}{[2\alpha(L-1)]^2}$$

We study the numerator:

$$\left\{[(1-L)[L + (\alpha + p)(1 - L)] + 2(L-1)p]\frac{1}{\sqrt{\Delta}} - (1-L)\right\} 2\alpha(L-1) - \left[\sqrt{\Delta} - [L + (\alpha + p)(1 - L)\right] 2(L-1)$$

$$\Leftrightarrow 2(L-1)\left\{\frac{\alpha(1-L)(L + (\alpha + p)(1 - L)}{\sqrt{\Delta}} + \frac{\alpha 2(L-1)p}{\sqrt{\Delta}} - \alpha(1-L)\right\} - \left[\sqrt{\Delta} - [L + (\alpha + p)(1 - L)\right] 2(L-1)$$

$$\Leftrightarrow 2(L-1)\left\{\frac{\alpha(1-L)(L + (\alpha + p)(1 - L) + 2\alpha(L-1)p - \alpha(1-L)\sqrt{\Delta} - \Delta + \sqrt{\Delta}[L + (\alpha + p)(1 - L)}{\sqrt{\Delta}}\right\}$$

Let's compute:

$$\alpha(1-L)(L+(\alpha+p)(1-L)+2\alpha(L-1)p-\alpha(1-L)\sqrt{\Delta}-\Delta+\sqrt{\Delta}[L+(\alpha+p)(1-L)$$

$$\Leftrightarrow \sqrt{\Delta}\,[L + p(1-L)] - 2\alpha(L-1)p + [L+(\alpha+p)(1-L)][\alpha(1-L) - L - (\alpha+p)(1-L)]$$

$$\Leftrightarrow [L + p(1-L)]\left[\sqrt{\Delta} - [L + (\alpha + p)(1 - L)]\right] - 2\alpha(L-1)p$$

We have:

$$[L + p(1-L)]\left[\sqrt{\Delta} - [L + (\alpha + p)(1 - L)]\right] - 2\alpha(L-1)p > 0$$

$$\Leftrightarrow p < [L + p(1-L)]\, q^*$$

Simulations show this inequation holds.
Finally:

$$\frac{\partial q^*}{\partial \alpha} = \frac{2(L-1)\left\{\frac{[L+p(1-L)]\left[\sqrt{\Delta}-[L+(\alpha+p)(1-L)]\right]-2\alpha(L-1)p}{\sqrt{\Delta}}\right\}}{[2\alpha(L-1)]^2} > 0$$

Since F is a strictly increasing function , when α rises, the right hand side is strictly decreasing. **Hence, the equilibrium price is a decreasing function of α.**

Proof of Proposition 2

Assuming a uniform distribution of initial endowments, we derive the equilibrium price from Proposition 1

$$p = 1 - \frac{\sqrt{\Delta} - [L + (\alpha + p)(1-L)]}{2\alpha(L-1)}$$

$$\Leftrightarrow (p-1)2\alpha(L-1) = -\sqrt{\Delta} + L + (\alpha + p)(1-L)$$

$$\Leftrightarrow [(1-p)2\alpha(L-1) + L + (\alpha + p)(1-L)]^2 = \Delta$$

By developing the LHS and replacing the RHS by (4), one gets:

$$[(1-p)2\alpha(L-1)]^2 + 2(1-p)2\alpha(L-1)\,[L + (\alpha+p)(1-L)] + [L + (\alpha+p)(1-L)]^2 = [L + (\alpha+p)(1-L)]^2 + 4\alpha(L-1)p$$

One eventually finds:

$$4\alpha(L-1)\left[p^2(L-1)(1+\alpha) + p[\alpha(1-L) - 2L] + L\right] = 0$$

which is equivalent to $p^2(L-1)(1+\alpha) + p[\alpha(1-L) - 2L] + L = 0$
We recognize once again a quadratic equation with discriminant:

$$\Delta = [\,\alpha(1-L) - 2L]^2 - 4L(L-1)(1+\alpha)$$

Using simulations, we show this discriminant is positive. Then we define two solutions for the price:

$$p_1 = \frac{2L + \alpha(L-1) - \sqrt{\Delta}}{2(L-1)(1+\alpha)}$$

$$p_2 = \frac{2L + \alpha(L-1) + \sqrt{\Delta}}{2(L-1)(1+\alpha)}$$

One can show that $p_2 > 1$ while $0 < p_1 < 1$, then the competitive equilibrium price in case of a uniform distribution of initial endowments is

$$p = \frac{2L + \alpha(L-1) - \sqrt{\Delta}}{2(L-1)(1+\alpha)}$$

Proof of Proposition 3

Using simulations with a uniform distribution of initial wealth, we find cases where the underreaction inequation does not hold.

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